Robust Control Design via Feedback Linearization

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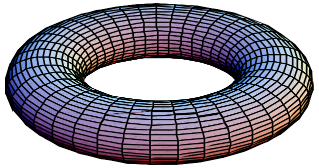

Outline

- Perturbations of feedback linearizable systems
- Triangular forms
- Lyapunov Redesign

Setup

$$\dot{x} = f(x) + g(x)u$$

$$y = h(x)$$

$$x \in R^n, u \in R, y \in R$$

$$f(x), g(x), h(x) \text{ smooth}$$

$$\dot{z} = Az + bv$$

$$y = cz$$

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Perturbations of SISO Feedback Linearizable Systems

$$\dot{x} = f(x) + \varphi(x) + [g(x) + \gamma(x)]u$$

$$y = h(x)$$

Assumption: the nominal system is feedback linearizable.

What happens when the nominal system transformation is applied to the actual system?

Matched Uncertainty, etc.

$$\mathcal{G}_i = \operatorname{span}\left\{ g, \ldots, ad_f^i g \right\}, 0 \le i \le n-1$$

Definition: Suppose the system is of relative degree *r*.
We say that the perturbation satisfies:

- **The *triangularity condition* if** $ad_\varphi \mathcal{G}_i \in \mathcal{G}_{i+1}, 0 \le i \le r-3$
- **The *strict triangularity condition* if** $ad_\varphi \mathcal{G}_i \in \mathcal{G}_{i+1}, 0 \le i \le r-2$
- **The *extended matching condition* if** $\varphi \in \mathcal{G}_1$
- **The *matching condition* if** $\varphi \in \mathcal{G}_0$

Triangular Forms ($r = n$)

- $triangularity \Rightarrow$

$$\dot{z}_i = z_{i+1} + \phi_i\left(z_1, \ldots, z_{i+1}\right), \quad 1 \le i \le n-1$$

$$\dot{z}_n = \alpha\left(x(z)\right) + \phi_n\left(z_1, \ldots, z_n\right) + \rho\left(x(z)\right)u$$

- $strict\ triangularity \Rightarrow$

$$\dot{z}_i = z_{i+1} + \phi_i\left(z_1, \ldots, z_i\right), \quad 1 \le i \le n-1$$

$$\dot{z}_n = \alpha\left(x(z)\right) + \phi_n\left(z_1, \ldots, z_n\right) + \rho\left(x(z)\right)u$$

- $extended\ matching \Rightarrow$

$$\dot{z}_i = z_{i+1}, \quad 1 \le i \le n-2$$

$$\dot{z}_{n-1} = \phi_{n-1}\left(z_1, \ldots, z_n\right)$$

$$\dot{z}_n = \alpha\left(x(z)\right) + \phi_n\left(z_1, \ldots, z_n\right) + \rho\left(x(z)\right)u$$

- $matching \Rightarrow$

$$\dot{z}_i = z_{i+1}, \quad 1 \le i \le n-1$$

$$\dot{z}_n = \alpha\left(x(z)\right) + \phi_n\left(z_1, \ldots, z_n\right) + \rho\left(x(z)\right)u$$

least restrictive

$\Downarrow$

most restrictive

Example

$$\dot{x} = \underbrace{\begin{bmatrix} x_2 \\ x_3 - x_1 - x_2 \\ x_4 \\ x_1 - x_3 - x_4/2 \end{bmatrix}}_{f} + \underbrace{\begin{bmatrix} 0 \\ -\Delta_1(x_1, x_2) \\ 0 \\ -\Delta_2(x_3, x_4) \end{bmatrix}}_{\phi} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{g} u$$

$$\mathcal{G}_0 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \mathcal{G}_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \mathcal{G}_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example Cont'd

$$ad_\phi \mathcal{G}_0 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial\Delta_2/\partial x_4 \end{bmatrix} \right\}, \; ad_\phi \mathcal{G}_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial\Delta_2/\partial x_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial\Delta_2/\partial x_4 \end{bmatrix} \right\},$$

$$ad_\phi \mathcal{G}_2 = \left\{ \begin{bmatrix} 0 \\ \partial\Delta_1/\partial x_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial\Delta_2/\partial x_3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \partial\Delta_2/\partial x_4 \end{bmatrix} \right\}$$

$$\Downarrow$$

strict triangularity

Example Cont'd

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 \\ z_3 &= -x_1 - x_2 + x_3 \\ z_4 &= x_1 - x_3 + x_4 \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= z_1 + z_2 + z_3 \\ x_4 &= z_2 + z_3 + z_4 \end{aligned}$$

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 - \Delta_1(z_1, z_2) \\ z_4 + \Delta_1(z_1, z_2) \\ \tfrac{1}{2}\left(-3z_2 - 5z_3 - 3z_4 - 2\Delta_2(z_1 + z_2 + z_3, z_2 + z_3 + z_4)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Lyapunov Redesign Process

- Begin with an exactly feedback linearizable nominal system with matched uncertainty
- Apply the nominal control $u^*(z)$ to actual system and test stability using a Lyapunov function
- Add a new component to nominal control, $u(z) = u^*(z) + \rho(z)^{-1}\mu(z)$ to enhance stability
- Choose $\mu(z)$ via Lyapunov design

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Problem Setup

- A linearizable system with matched uncertainty is transformable to:

$$(*)\quad \dot{z} = Az + E\left[\alpha(z) + \Delta(z,u,t) + \rho(z)u\right]$$

- Choose a control based on nominal system:

$$u^*(t) = \rho^{-1}(z)\left(-\alpha(z) + Kz\right), \quad with\left(A + EK\right) stable$$

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Nominal Closed Loop System

$$(**) \quad \dot{z} = (A + EK)z$$

$$(A + EK)\ stable \Rightarrow Lyapunov\ \ function$$

$$V(z) = z^T Pz, \quad \dot{V}(z) = -z^T Qz - \|z\|^2$$

$$P(A + EK) + (A + EK)^T P = -Q - I, \quad Q = Q^T > 0$$

along trajectories of (**)

Lyapunov Equation

Actual Closed Loop System

- Apply *u** to actual system (*). Try *V(z)*

$$\dot{V}(z) = -z^T Q z - \|z\|^2 + 2 z^T P E \Delta$$

- Assume

$$\|\Delta\| < \gamma \|z\|, \quad \gamma \geq 0$$

$$\dot{V} \leq -\left(\lambda_{\min}(Q) + 1\right)\|z\|^2 + 2\gamma \|PE\| \|z\|^2$$

$$\Downarrow$$

$$\textit{stability if } \gamma < \left(\lambda_{\min}(Q) + 1\right) / 2\|PE\|$$

This implies some inherent robustness

Redesign (we can do more) ~ 1

$$u = u^* + \rho^{-1}\mu$$

- Assume the uncertainty satisfies:

$$\Delta(0,0,t) = 0,\ \forall t$$

$$\left\|\Delta\left(z, u^* + \rho^{-1}\mu, t\right)\right\| \le \sigma(z)\|z\| + k\|\mu\|,\ 0 \le k < 1,\ \sigma\ smooth$$

- The actual closed loop dynamics are:

$$\dot{z} = (A + EK)z + E\left(\mu + \Delta\left(z, u^* + \rho^{-1}\mu, t\right)\right)$$

- The time derivative of V along trajectories is

$$\dot{V} = -z^T Qz - \|z\|^2 + 2z^T PE(\mu + \Delta)$$

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Redesign ~ 2

- Set $w^T := z^T PE$ and try to achieve $-\|z\|^2 + w^T(\mu + \Delta) \le 0$
- Notice that

$$w^T\mu + w^T\Delta \le w^T\mu + \|w\|\|\Delta\| \le w^T\mu + \|w\|\left[\sigma(z)\|z\| + k\|\mu\|\right]$$

- For a smooth control, set $\mu := -w\kappa, \kappa(z) > 0$

$$-\|z\|^2 + w^T\mu + w^T\Delta \le -\|w\|^2\kappa(1-k) + \|w\|\sigma(z)\|z\| - \|z\|^2$$

$$\kappa = \frac{1}{4(1-k)}\sigma^2 \Rightarrow$$

$$-\|z\|^2 + w^T\mu + w^T\Delta \le -\tfrac{1}{4}\|w\|^2\sigma^2 + \sigma\|w\|\|z\| - \|z\|^2 = -\left(\tfrac{1}{2}\|w\|\sigma - \|z\|\right)^2$$

Redesign ~ 3

- So any $\kappa > \frac{1}{4(1-k)}\sigma^2$ will do

- In particular

$$\mu = -\left(\sigma_0 + \frac{\sigma^2(z)}{4(1-k)}\right)E^T P z, \quad \sigma_0 > 0$$

Redesign ~ 4

- There are other possibilities, recall

$$\dot{V} = -z^T Q z - \|z\|^2 + w^T \left(\mu + \Delta\right)$$

- And

$$w^T \mu + w^T \Delta \le w^T \mu + \|w\| \left[\sigma(z)\|z\| + k\|\mu\|\right]$$

- So choose

$$\mu = -\frac{\eta(z)}{1-k}\frac{w}{\|w\|}, \quad \eta(z) > \sigma(z)\|z\|$$

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_2\\ -0.2x_2+x_1^3/2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\left[u+\kappa x_1^3+au\right]$$

$$\kappa\in[0,1],\quad a\in[-0.1,0.1]$$

$$\alpha=-0.1x_2+x_1+x_1^3/2,\ \rho=1$$

$$u^*=\rho^{-1}\left(-\alpha+\begin{bmatrix}-1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)=-2x_1-1.9x_2-x_1^3/2$$

$$\Delta=\kappa x_1^3+a\left(-2x_1-\frac{x_1^3}{2}-1.9x_1\right)+a\mu$$

$$\sigma^2=0.0761+1.1025x_1^4,\ k=a$$

$$u=u^*+\mu=-5.02114x_1-\frac{x_1^3}{2}-0.30625x_1^5-4.92114x_2-0.30625x_1^4x_2$$

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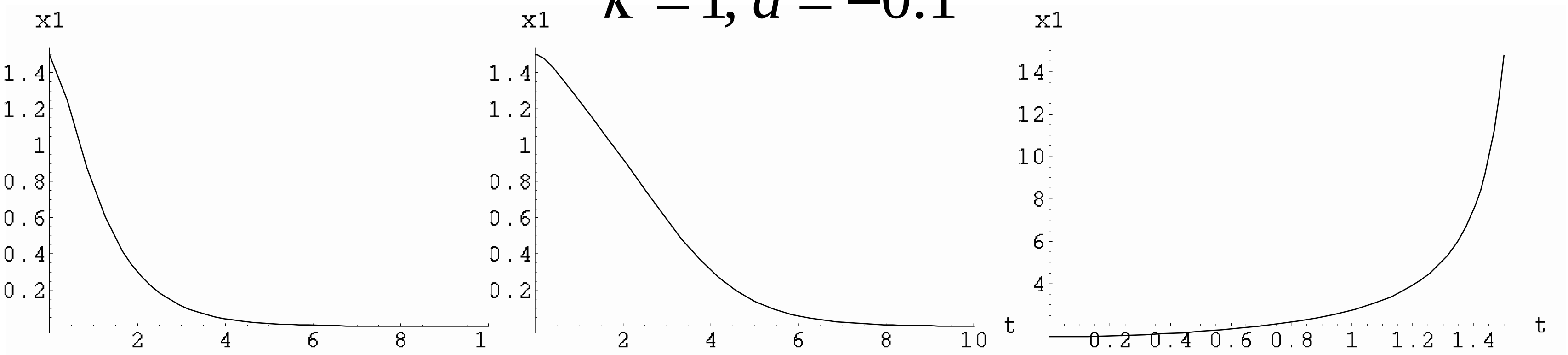
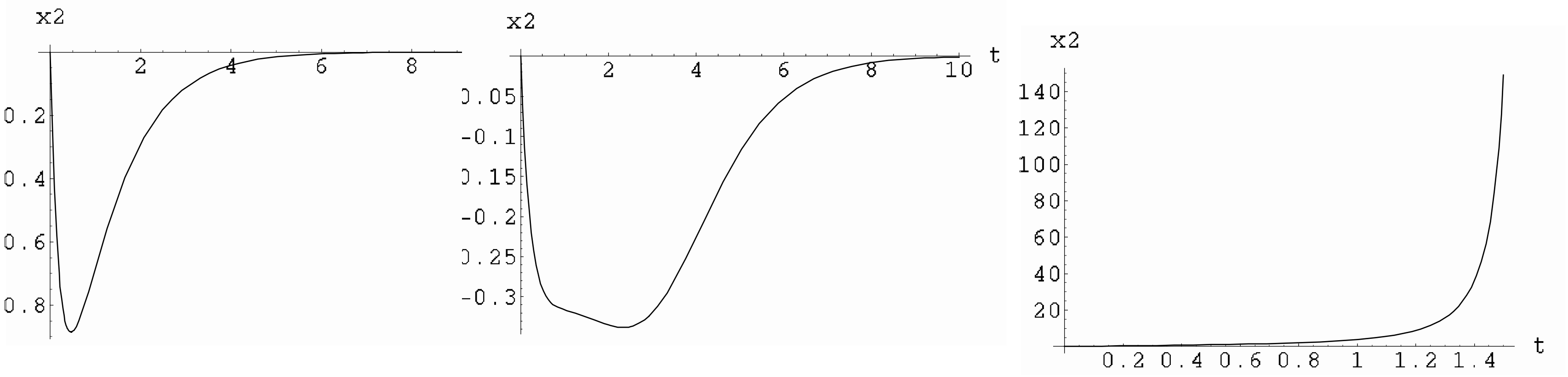
Example ~ Results

$$\kappa = 1, a = -0.1$$



nom system/nom control actual system/robust control actual system/nom control



Setup for Backstepping

- SISO system
- $r = n$
- strict triangularity assumption

$$\dot{x}_i = x_{i+1} + \Delta_i\left(x_1, \ldots, x_i, t\right), \quad 1 \le i \le n-1$$

$$\dot{x}_n = \alpha(x) + \rho(x)u + \Delta_n(x,t)$$

$$\det \rho(0) \neq 0$$

$$\Delta(0,t) = 0, \left|\Delta_i\left(X_i, t\right)\right| \le \sigma_i\left(X_i\right)\left\|X_i\right\|, \sigma_i \ge 0$$

$$X_i = \left\{x_1 \quad \cdots \quad x_i\right\}$$

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Step 1

$$\begin{aligned}\dot{x}_1 &= v_1 + \Delta_1(x_1, t) \\ y_1 &= x_1\end{aligned} \quad \Rightarrow \dot{y}_1 = v_1 + \bar{\Delta}_1(y_1, t) \coloneqq f_1(y_1)$$

$$V_1 \coloneqq \tfrac{1}{2} y_1^2$$

$$L_{f_1} V_1 = y_1\left(v_1 + \bar{\Delta}_1(y_1, t)\right)$$

$$choose: v_1 = -k_1 y_1 - \kappa_1(y_1) y_1$$

$$\Downarrow$$

$$L_{f_1} V_1 = -k_1 y_1^2 - \kappa_1(y_1) y_1^2 + y_1 \bar{\Delta}_1$$

$$take: \kappa_1(y_1) > \tfrac{1}{4} \sigma_1^2(x_1),\ x_1 \neq 0$$

$$\Downarrow$$

$$L_{f_1} V_1 = \leq -(k_1 - 1) y_1^2$$

Step 2

$$\dot{x}_1 = x_2 + \Delta_1(x_1, t)$$
$$\dot{x}_2 = v_2 + \Delta_2(x_1, x_2, t)$$
$$y_2 = x_2 - v_1(x_1)$$
$$(x_1, x_2) \mapsto (y_1, y_2)$$
$$\Downarrow$$
$$\left.\begin{array}{l} \dot{y}_1 = v_1(y_1) + y_2 + \bar{\Delta}_1(y_1, t) \\ \dot{y}_2 = v_2 + \bar{\Delta}_2(y_1, y_2, t) \end{array}\right. := f_1(y_1, y_2)$$
$$define : V_2 = V_1 + \tfrac{1}{2} y_2^2$$
$$choose : v_2 = -y_1 - k_2 y_2 - \kappa_2(y_1, y_2) y_2$$
$$\kappa_2(y_1, y_2) > \tfrac{1}{4} \bar{\sigma}_2^2(y_1, y_2) \Rightarrow L_{f_2} V_2 \le -(k_1 - 2) y_1^2 - (k_2 - 1) y_2^2$$